Stopping distance for high energy jets in weakly-coupled quark-gluon plasmas

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Abstract

We derive a simple formula for the stopping distance for a high-energy quark traveling through a weakly-coupled quark gluon plasma. The result is given to next-to-leading-order in an expansion in inverse logarithms $\ln(E/T)$, where T is the temperature of the plasma. We also define a stopping distance for gluons and give a leading-log result. Discussion of stopping distance has a theoretical advantage over discussion of energy loss rates in that stopping distances can be generalized to the case of strong coupling, where one may not speak of individual partons.

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I. INTRODUCTION AND RESULTS

There has long been interest in calculating how a fast moving quark or gluon looses energy when traveling through a quark-gluon plasma. One of the simpler versions of this problem is to calculate energy loss in the case of a spatially-infinite equilibrium plasma (i.e. a plasma whose size is large compared to the stopping distance). For weakly-coupled plasmas, energy loss is dominated by hard bremsstrahlung and pair production in the high-energy limit $E \gg T$. Results for the rates of these processes at leading order in α_s have been known for some time, as we'll review in a moment. The rate is parametrically [1, 2, 3, 4, 5]

$$\Gamma \sim \alpha_{\rm s}^2 T \sqrt{\frac{\ln(E/T)}{E/T}} \tag{1.1}$$

for hard bremsstrahlung and pair production at high energy (with some caveats on the range of applicability to be discussed later). The roughly $E^{-1/2}$ decrease of this rate with energy is a result of the Landau-Pomeranchuk-Migdal (LPM) effect [6, 7], which arises from interference between bremsstrahlung (or pair production) associated with multiple collisions of the particle with the plasma. If there were no LPM effect, then the high-energy rate would have been approximately independent of E.

From the rate of bremsstrahlung and pair production, one can figure out how far a high-energy particle travels before it "stops" in the plasma, by which we mean that the particle loses enough energy to reach $E \sim T$ and so equilibrate with the plasma. Consider the energy loss of a quark. Crudely, if the quark loses roughly half its energy in the first hard bremsstrahlung, and roughly half in the next, and so forth, then the roughly $E^{-1/2}$ behavior of the rate (1.1) suggests that the stopping distance is parametrically of order

$$\begin{aligned}
\ell_{\rm stop} &\sim \frac{1}{\Gamma(E)} + \frac{1}{\Gamma(E/2)} + \frac{1}{\Gamma(E/4)} + \frac{1}{\Gamma(E/8)} + \cdots \\
&\sim \frac{1}{\Gamma(E)} \left(1 + \frac{1}{2^{1/2}} + \frac{1}{2} + \frac{1}{2^{3/2}} + \cdots \right) \\
&\sim \frac{1}{\Gamma(E)} \sim \frac{1}{\alpha_{\rm s}^2 T} \sqrt{\frac{E/T}{\ln(E/T)}} \, . \tag{1.2}
\end{aligned}$$

In weak coupling, the $E^{-1/2}$ fall-off of the rate is due to the LPM effect. It's interesting to ascertain whether anything similar happens in plasmas that are not weakly coupled. Fortunately, there are certain strongly-coupled gauge theories where people have, using AdS/CFT duality, calculated results related to the slowing down of high-momentum particles. We cannot directly discuss bremsstrahlung or pair production rates at strong coupling, however, because bremsstrahlung and pair production implicitly refer to individual quanta and so are intrinsically perturbative concepts. But one can generalize the idea of stopping distances to non-perturbative situations, as nicely explained by Chesler, Jensen, Karch, and Yaffe [8], as we shall review later. In the strongly-coupled gauge theories that have been studied, stopping distances at high energy behave like $E^{1/3}$ [8, 9, 10] rather than $E^{1/2}/\sqrt{\ln E}$.

Because stopping distances, unlike bremsstrahlung rates, can be generalized to strongly-coupled situations, it's interesting to know how to calculate them in weakly-coupled situations. In this paper, we derive simple results for various stopping distances in QCD in the high-energy limit $\ln(E/T) \gg 1$, working to leading order in coupling $\alpha_{\rm s}$. Though we assume

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N_f	a	b	c	$a_g^{(e)}$	$a_q^{(e)}$
QCD 0	—	—	0.736866	5.26606	—
2	2.06480	0.092141	0.432769	5.98760	3.61355
3	2.23024	0.105282	0.345885	6.27235	3.87544
4	2.38423	0.115487	0.280721	6.52525	4.11632
5	2.52886	0.123314	0.230039	6.75427	4.34031
6	2.66565	0.129246	0.189492	6.96480	4.55038
QED 1	0.30216	3.16393	-0.175423	0.11121	0.12072

TABLE I: Coefficients a,b,c for the quark-number stopping distance (1.3) in the case $Q_\perp \gg T$ (parametrically $E \gg T^4/\hat{q}$ or equivalently $E \gg T/\alpha_s^2 \ln(\alpha_s^{-1})$). Also shown are the leading-log coefficients $a_g^{(e)}$ and $a_q^{(e)}$ for the gluon and quark energy stopping distances defined in Sec. II B. Note that larger values of a correspond to shorter stopping distances in the high-energy limit.

N_f	a	b	c	$a_g^{(e)}$	$a_q^{(e)}$
QCD 0	—	—	0	6.16024	—
2	2.27727	0.533221	0	6.60372	3.98538
3	2.41541	0.473974	0	6.79311	4.19719
4	2.54606	0.426577	0	6.96816	4.39572
5	2.67033	0.387797	0	7.13213	4.58313
6	2.78907	0.355480	0	7.28728	4.76106
QED 1	0.28861	2.92883	0	0.10622	0.11531

TABLE II: Coefficients for the stopping distance in the case $Q_\perp \ll T$ (parametrically $T \ll E \ll T^4/\hat{q} \sim T/\alpha_s^2 \ln(\alpha_s^{-1})$).

$\ln(E/T) \gg 1$, we will formally assume that α_s is small enough that $\alpha_s \ln(E/T) \ll 1$ and so, for instance, we ignore the running of the coupling α_s. (We will say a few words later about how to plausibly modify the answer to include running coupling effects.)

Our most thorough calculation is of the stopping distance for a quark, which we compute to next-to-leading-logarithmic order (NLLO) in $\ln(E/T)$. Our results can be summarized as

$$\ell_{\mathrm{stop,q}} \simeq \frac{1}{a\alpha_s^2 T}\sqrt{\frac{E}{TL}}, \tag{1.3a}$$

where the logarithm L is determined by self-consistent solution of the equation

$$L = \ln\left(\frac{bEL}{\alpha_s^c T}\right) \tag{1.3b}$$

and a, b, and c are numerical constants. These constants are given in Table I in the large E limit of $E \gg T/\alpha_s^2 \ln(\alpha_s^{-1})$, or in Table II in the case $T \ll E \ll T/\alpha_s^2 \ln(\alpha_s^{-1})$ of smaller (but still formally large) E. The latter case has $c = 0$ in (1.3b) and so corresponds to the simple parametric estimate (1.2) based on (1.1).

Later, we will also give a definition of a stopping distance for gluons that can be generalized to non-perturbative gauge theories, and we will give a result to leading-log order for that distance in the case of weak coupling.

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At the moment, we do not have a precise weakly-coupled result for the types of super-symmetric gauge theories in which the strongly-coupled limit has been studied. The answer will be in the form (1.3) for weak coupling, but we leave the calculation of the coefficients a, b, and c in this case for future work.

Throughout this paper, we will assume that the baryon chemical potential of the quark-gluon plasma is negligible compared to its temperature.

In section II, we review Chesler et al.'s definition of a stopping distance for quarks, which we call the quark number stopping distance, and then we discuss a natural generalization for discussing gluons, which introduces what we will call the quark and gluon energy stopping distances. Section III derives results for these various stopping distances to leading-log order, which is enough to determine the coefficients a and c in our result (1.3). Section IV then continues to next-to-leading log order for the quark number stopping distance, giving the remaining coefficient b. As a test of our NLLO result, we compare it in section V to numerical simulations of energy loss that make no approximation regarding the size of $\ln(E/T)$ but instead use full weak-coupling results for the bremsstrahlung rate. We verify that our NLLO result (1.3) correctly reproduces the large E behavior. Finally, in section VI, we discuss how one might modify our result to account for running of $\alpha_{\rm s}$.

II. DEFINING STOPPING DISTANCES

A. Quark Number Stopping

To explain Chesler et al.'s definition of a stopping distance for quarks, we first focus on the case of weak coupling, where we can talk about individual particles. We gave a crude estimate of the stopping distance in (1.2), but we should consider that an individual quark doesn't really stop in a weakly-coupled plasma. Once its energy drops to $E \sim T$, then the quark is just as likely to gain energy from its interactions with the plasma as lose energy. A cartoon of an individual high-energy quark's trajectory is shown in Fig. 1a. At first, the quark moves in a nearly straight line as it loses energy. But once its energy falls to be $\sim T$, it then random walks through the plasma just like any other equilibrated quark. Suppose we repeat this thought experiment over and over again in order to determine the probability *distribution* of the quark's position as a function of time. A cartoon of the time evolution of this distribution is shown in Fig. 1b. Once the quark equilibrates, the center of the distribution stops moving, and the distribution merely spreads in size due to diffusion. We can then define the stopping distance of a quark as the distance that the center of the probability distribution moves between t=0 and t=∞.

So far, we have been speaking perturbatively about the position of the quark. The probability distribution of quark positions easily generalizes to the non-perturbative case: simply refer instead to the probability distribution for total quark number in the system, since the quark number operator is defined non-perturbatively. But so far we've specified our initial state perturbatively when we said it was a single high-energy quark moving through the quark-gluon plasma. Chesler et al. realized that this can also be generalized to the non-perturbative case. Consider all possible initial states that (i) differ from thermal equilibrium in a localized region of space, (ii) contain total energy E in that region in excess of the thermal background, and (iii) contain total quark number one in that region. Measure the stopping distance for that initial state. Then define the "quark number stopping distance" as the maximum stopping distance, taken over all such choices of initial state.
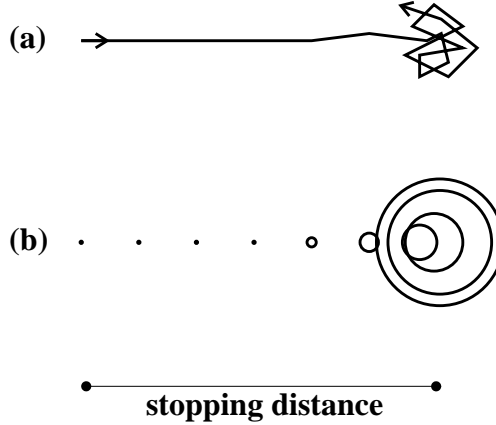
FIG. 1: The stopping of a high-energy quark represented by (a) the path of an individual quark, and (b) the motion of the center of the spreading quark number probability distribution. In (b), the quark number probability distribution at each successive time is represented by a *single* contour. As time progresses, the center of the contour moves from left to right, slowing down with time, and the diameter of the contour increases.

Let's understand how this definition would work out in the weakly-coupled case. In our earlier discussion, our initial state was a single quark carrying the total energy E. What if we had chosen a different initial state with the same energy and quark number? Consider, for instance, an initial state composed of a quark of energy $E/2$ plus a gluon of energy $E/2$. The quark number stopping distance doesn't care about what happens to the gluon and would just depend on how long it took the energy $E/2$ quark to stop. But according to (1.2), that is roughly $1/\sqrt{2}$ shorter time than if the entire energy E had been carried by a single quark. So the initial state that gives the *maximum* stopping distance is indeed the one we want in the weakly-coupled case: all the energy carried by a single quark.

The definition of stopping time just given implicitly assumed that both the position and energy of the initial state are well defined. Due to the uncertainty principle, however, the location of the initial position will naturally be ambiguous by an amount of order $1/E$. So the definition of stopping distance implicitly assumes the limit that $\ell_{\mathrm{stop}} \gg 1/E$.

In the case of $\mathcal{N}{=}4$ large-N_{c} supersymmetric Yang Mills theory with a massless $\mathcal{N} = 2$ fundamental-charge "quark" hypermultiplet, Chesler et al. find (with their non-perturbative definition of the stopping distance)

$$\ell_{\mathrm{stop,q}} = \frac{\mathcal{C}}{T} \left(\frac{E}{T\sqrt{\lambda}} \right)^{1/3} \tag{2.1}$$

in the $\lambda \equiv N_{\mathrm{c}} g_{\mathrm{s}}^2 \to \infty$ limit, with

$$\mathcal{C} \approx 0.5 \,. \tag{2.2}$$

B. Gluon Energy Stopping

Even in weak coupling, there is some difficulty in defining exactly what one means by the stopping distance of a gluon. For quarks, it's always clear what the energy of the quark

is after a $q \to qg$ splitting process. For gluons, we can have $g \to gg$ or $g \to q\bar{q}$, and we have to decide which particle to follow. One possibility would be to follow, at each splitting, the final particle that has the highest energy. This definition does not have any obvious generalization to strong coupling. But we can instead use a definition like the previous one for quark number stopping distance. For the initial state, we now require quark number zero. Then, for the stopping distance, we measure the distance traveled before the center of the probability distribution for *energy* in excess of equilibrium (rather than quark number) stops. We will call this the "gluon energy stopping distance."

For a gluon in a weakly-coupled plasma, this definition can be translated as follows. Consider the splitting of an initial high-energy gluon, moving in the z direction, which cascades through splitting into N particles, which are "stopped" at positions z_1, z_2, ..., z_N relative to the initial position of the initial gluon. Here stopped means that the individual energies E_i are of order T. The gluon energy stopping distance is the energy-weighted average of these stopping positions:

$$\ell_{\text{stop,g}}^{(\text{energy})} \simeq \frac{\sum_i E_i z_i}{E} \,. \tag{2.3}$$

This simple translation is fuzzy when it comes to deciding exactly when a final parton has low enough energy to be considered "stopped." Is it $E_i \leq T$ or $E_i \leq 2T$ or ...? However, because the time scale $(\alpha^2 T)^{-1}$ for particles with energy of order T to significantly interact is so much faster than the time scale (1.2) for the initial gluon to lose significant energy, this ambiguity only involves corrections to the calculation of the stopping distance that are suppressed by a factor of $\sqrt{T/E}$ (up to logarithms). We can ignore these corrections if we are just interested in $E \gg T$. Note that, when $E_i \sim T$, we should also be thinking about collisional energy loss, which is then competitive with bremsstrahlung and pair production. But the effects of collisional loss on the stopping distance will similarly be suppressed by a factor of $\sqrt{T/E}$.

III.   LEADING LOG

A.   Quark Number Stopping: General Analysis

We'll begin with a leading-log calculation of the quark number stopping distance, which is fairly simple. To motivate the approximation, consider the crude, heuristic estimate made in (1.2). At leading-log order, we can ignore the difference between a $\ln(E/T)$ appearing in $\Gamma(E)$, a $\ln(E/2T)$ appearing in $\Gamma(E/2)$, and a $\ln(E/4T)$ appearing in $\Gamma(E/4)$. Since the series in (1.2) converges sufficiently rapidly (i.e. geometrically), it is sufficient at leading-log order to replace the logarithm in the bremsstrahlung rate (1.1) by a constant:

$$\Gamma(E) \sim \alpha_{\text{s}}^2 T \sqrt{\frac{\ln(E_0/T)}{E/T}} \tag{3.1}$$

where E_0 is the initial energy of the quark. In this approximation, the E dependence of $\Gamma(E)$ is simply $E^{-1/2}$, without any logarithm. That simplification will be the key to a simple solution for the stopping distance in what follows.

In more detail, the bremsstrahlung rate is a function of the momentum fraction $x = \omega/E$ of the emitted gluon of frequency ω. (For hard bremsstrahlung in the large E limit, we can

ignore the thermal mass of the gluon.) In the leading-log approximation, we can write

$$\frac{d\Gamma(E)}{dx} \simeq \left(\frac{E_0}{E}\right)^{1/2} \frac{d\Gamma(E_0)}{dx} \equiv E^{-1/2} \frac{d\tilde{\Gamma}}{dx}, \tag{3.2}$$

where there is an implicit logarithmic dependence of $d\tilde{\Gamma}/dx$ on E_0. We will review the specific leading-log formula for $q \to gq$ bremsstrahlung later. But for now we can proceed with the analysis in a general way, based simply on the scaling (3.2).

Let $\ell_q(E)$ be the quark number stopping distance as a function of E. We can formally write the following self-consistent equation for ℓ_q:

$$\ell_q(E) = \frac{1}{\Gamma_{q \to gq}(E)} + \int_0^1 dx \, \frac{d\Gamma_{q \to gq}(E,x)/dx}{\Gamma_{q \to gq}(E)} \, \ell_q\big((1-x)E\big). \tag{3.3}$$

The first term on the right-hand side is the average distance before the first bremsstrahlung. That first bremsstrahlung splits the quark into a nearly-collinear gluon with energy xE and quark with energy $(1-x)E$. The second term on the right-hand side of (3.3) is the remainder of the stopping distance after the first bremsstrahlung. This is just $\ell_q((1-x)E)$ weighted by the probability $(d\Gamma/dx)/\Gamma$ that the first bremsstrahlung had gluon momentum fraction x. To find the stopping distance, we just need to solve this equation in leading-log approximation.

Note that (3.3) ignores what happens to the radiated gluons. Suppose a bremsstrahlung gluon later splits into a $q\bar{q}$ pair with the new quark carrying more energy than the new anti-quark. The subsequent cascading of that pair will then affect the distribution of quark charge at the end of the cascade process, and so will affect where the center of the total quark number distribution stops. However, it's just as likely that the gluon would have instead split into a $q\bar{q}$ pair in which the new anti-quark carries more energy, and this will produce an opposite effect on the location of the center of the final quark number distribution. Because of charge symmetry, we can ignore what happens to the gluons when calculating the quark number stopping distance. We dwell on this point because the situation will be different when we later compute the gluon energy stopping distance.

The reason we referred to (3.3) as "formal" is that the total bremsstrahlung rate

$$\Gamma_{q \to gq}(E) = \int_0^1 dx \, \frac{d\Gamma_{q \to gq}(E,x)}{dx} \tag{3.4}$$

is infrared ($x \to 0$) divergent in leading-log approximation. You can imagine temporarily imposing some sort of infrared regulator. But it is easy to manipulate (3.3) to remove the need for such a regulator by multiplying both sides by $\Gamma_{q \to gq}(E)$ and rearranging terms to rewrite the equation as

$$\int_0^1 dx \, \frac{d\Gamma_{q \to gq}(E,x)}{dx} \big[\ell_q(E) - \ell_q\big((1-x)E\big)\big] = 1. \tag{3.5}$$

The fact that the factor in brackets vanishes as $x \to 0$ turns out to be sufficient to make the x integration in (3.5) infrared convergent.

Now make the leading-log approximation by substituting (3.2) into (3.5) to get

$$E^{-1/2} \int_0^1 dx \, \frac{d\tilde{\Gamma}_{q \to gq}(x)}{dx} \big[\ell_q(E) - \ell_q\big((1-x)E\big)\big] = 1. \tag{3.6}$$

We can solve this equation by writing

$$\ell_{\mathrm{q}}(E) = \frac{E^{1/2}}{A_{\mathrm{q}}}, \tag{3.7}$$

where A_{q} is a constant. Then the factors of E drop out of (3.6), leaving

$$A_{\mathrm{q}} = \int_0^1 dx \, \frac{d\tilde{\Gamma}_{q \to gq}(x)}{dx} \left[1 - (1-x)^{1/2}\right]. \tag{3.8}$$

It only remains to plug in the detailed leading-log formula for $d\tilde{\Gamma}_{q \to gq}/dx$ in leading-log approximation. We'll leave that to Sec. III C.

B. Gluon Energy Stopping: General Analysis

For the gluon energy stopping distance $\ell_{\mathrm{g}}^{(\mathrm{e})}(E)$ given in weak coupling (and high energy) by (2.3), we can write down an equation similar to (3.3). One difference is that we now have to account for both of the high-energy particles present after the first splitting, because they both carry energy. To warm up, first consider pure Yang Mills gauge theory with no quarks. Then the only relevant splitting process would be $g \to gg$, and the equation for the gluon energy stopping distance would be

$$\ell_{\mathrm{g}}^{(\mathrm{e})}(E) = \frac{1}{\Gamma_{g \to gg}(E)} + \frac{1}{2} \int_0^1 dx \frac{d\Gamma_{g \to gg}(E,x)/dx}{\Gamma_{g \to gg}(E)} \left[x \, \ell_{\mathrm{g}}^{(\mathrm{e})}(xE) + (1-x) \, \ell_{\mathrm{g}}^{(\mathrm{e})}\big((1-x)E\big)\right]. \tag{3.9}$$

The new feature to this equation is that the energy stopping distances $\ell_{\mathrm{g}}^{(\mathrm{e})}(xE)$ and $\ell_{\mathrm{g}}^{(\mathrm{e})}\big((1-x)E\big)$ of the two gluons after the first splitting are averaged, weighted by their energies, as required by (2.3). (An energy-weighted average of the energy stopping distances of these two gluons is the same thing as an energy-weighted average of the positions of all their stopped descendants.) The overall factor of $\frac{1}{2}$ in front of the integral in (3.9) is to avoid double counting states of the final two, identical gluons.

Once quarks are introduced, there is the added complication that the energy carried by a gluon could be converted into energy carried by quarks via $g \to q\bar{q}$. To find the gluon energy stopping distance $\ell_{\mathrm{g}}^{(\mathrm{e})}(E)$, we will therefore need to also introduce the quark energy stopping distance $\ell_{\mathrm{q}}^{(\mathrm{e})}(E)$, which is given in weak coupling by (2.3) but in the case where the initial particle is a quark (or anti-quark). The basic equation (3.9) for $\ell_{\mathrm{g}}^{(\mathrm{e})}$ now becomes

$$\ell_{\mathrm{g}}^{(\mathrm{e})}(E) = \frac{1}{\Gamma_{\mathrm{g}}(E)} + \int_0^1 dx \Bigg\{ \frac{1}{2} \frac{d\Gamma_{g \to gg}(E,x)/dx}{\Gamma_{\mathrm{g}}(E)} \left[x \, \ell_{\mathrm{g}}^{(\mathrm{e})}(xE) + (1-x) \, \ell_{\mathrm{g}}^{(\mathrm{e})}\big((1-x)E\big)\right]$$
$$+ \frac{d\Gamma_{g \to q\bar{q}}(E,x)/dx}{\Gamma_{\mathrm{g}}(E)} \left[x \, \ell_{\mathrm{q}}^{(\mathrm{e})}(xE) + (1-x) \, \ell_{\mathrm{q}}^{(\mathrm{e})}\big((1-x)E\big)\right] \Bigg\}, \tag{3.10}$$

where

$$\Gamma_{\mathrm{g}} \equiv \Gamma_{g \to gg} + \Gamma_{g \to q\bar{q}} = \frac{1}{2} \int_0^1 dx \, \frac{d\Gamma_{g \to gg}}{dx} + \int_0^1 dx \, \frac{d\Gamma_{g \to q\bar{q}}}{dx} \tag{3.11}$$

is the total rate for a gluon to split by either bremsstrahlung or pair production. We now need the corresponding equation for the quark energy stopping distance:

$$\ell_{\mathrm{q}}^{(\mathrm{e})}(E) = \frac{1}{\Gamma_{q \to gq}(E)} + \int_0^1 dx \, \frac{d\Gamma_{q \to gq}(E,x)/dx}{\Gamma_{q \to gq}(E)} \left[x \, \ell_{\mathrm{g}}^{(\mathrm{e})}(xE) + (1-x) \, \ell_{\mathrm{q}}^{(\mathrm{e})}\big((1-x)E\big) \right]. \quad (3.12)$$

To solve this coupled system of equations, we proceed as before. First eliminate the issue of infrared divergences by rewriting them as

$$\int_0^1 dx \bigg\{ \frac{1}{2} \frac{d\Gamma_{g \to gg}(E,x)}{dx} \left[\ell_{\mathrm{g}}^{(\mathrm{e})}(E) - x \, \ell_{\mathrm{g}}^{(\mathrm{e})}(xE) - (1-x) \, \ell_{\mathrm{g}}^{(\mathrm{e})}\big((1-x)E\big) \right]$$
$$+ \frac{d\Gamma_{g \to q\bar{q}}(E,x)}{dx} \left[\ell_{\mathrm{g}}^{(\mathrm{e})}(E) - x \, \ell_{\mathrm{q}}^{(\mathrm{e})}(xE) - (1-x) \, \ell_{\mathrm{q}}^{(\mathrm{e})}\big((1-x)E\big) \right] \bigg\} = 1, \quad (3.13)$$

$$\int_0^1 dx \, \frac{d\Gamma_{q \to gq}(E,x)}{dx} \left[\ell_{\mathrm{q}}^{(\mathrm{e})}(E) - x \, \ell_{\mathrm{g}}^{(\mathrm{e})}(xE) - (1-x) \, \ell_{\mathrm{q}}^{(\mathrm{e})}\big((1-x)E\big) \right] = 1. \quad (3.14)$$

Then we make the leading-log approximation (3.2) and write

$$\ell_{\mathrm{g}}^{(\mathrm{e})}(E) = \frac{E^{1/2}}{A_{\mathrm{g}}^{(\mathrm{e})}}, \qquad \ell_{\mathrm{q}}^{(\mathrm{e})}(E) = \frac{E^{1/2}}{A_{\mathrm{q}}^{(\mathrm{e})}}, \quad (3.15)$$

to obtain coupled algebraic equations for $1/A_{\mathrm{g}}^{(\mathrm{e})}$ and $1/A_{\mathrm{q}}^{(\mathrm{e})}$. The solution is

$$A_{\mathrm{g}}^{(\mathrm{e})} = \frac{M_{\mathrm{gg}} M_{\mathrm{qq}} - M_{\mathrm{gq}} M_{\mathrm{qg}}}{M_{\mathrm{qq}} - M_{\mathrm{gq}}}, \qquad A_{\mathrm{q}}^{(\mathrm{e})} = \frac{M_{\mathrm{gg}} M_{\mathrm{qq}} - M_{\mathrm{gq}} M_{\mathrm{qg}}}{M_{\mathrm{gg}} - M_{\mathrm{qg}}}, \quad (3.16a)$$

where

$$M_{\mathrm{gg}} = \int_0^1 dx \bigg\{ \frac{1}{2} \frac{d\tilde{\Gamma}_{g \to gg}(x)}{dx} \left[1 - x^{3/2} - (1-x)^{3/2} \right] + \frac{d\tilde{\Gamma}_{g \to q\bar{q}}(E,x)}{dx} \bigg\}, \quad (3.16b)$$

$$M_{\mathrm{gq}} = \int_0^1 dx \, \frac{d\tilde{\Gamma}_{g \to q\bar{q}}(x)}{dx} \left[-x^{3/2} - (1-x)^{3/2} \right], \quad (3.16c)$$

$$M_{\mathrm{qg}} = \int_0^1 dx \, \frac{d\tilde{\Gamma}_{q \to gq}(x)}{dx} \left[-x^{3/2} \right], \quad (3.16d)$$

$$M_{\mathrm{qq}} = \int_0^1 dx \, \frac{d\tilde{\Gamma}_{q \to gq}(x)}{dx} \left[1 - (1-x)^{3/2} \right]. \quad (3.16e)$$

C. Leading Log Details

1. Bremsstrahlung and pair production rates

Following notation similar to Ref. [12], the leading-log splitting rates are

$$\frac{d\Gamma_{a \to bc}}{dx} = \frac{\alpha \, \mu_{a \to bc}^2 \, P_{a \to b}(x)}{4\pi \sqrt{2} \, x(1-x) E} \quad (3.17)$$

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where, at leading-log order,

$$\mu_{a\to bc}^2 \simeq \Big\{ 4x(1-x)E\big[(-C_a + C_b + C_c) + (C_a - C_b + C_c)x^2$$
$$+ (C_a + C_b - C_c)(1-x)^2\big]\hat{\bar{q}}(Q_{\perp 0})\Big\}^{1/2}. \quad (3.18)$$

Here $\hat{\bar{q}}(\Lambda)$ is proportional to the average squared transverse momentum $Q_\perp^2$ that a high-energy particle picks up per unit length while traveling through the plasma. We'll discuss the formula shortly. In the present context, the definition of $\hat{\bar{q}}(\Lambda)$ is UV-regulated by imposing an ultra-violet cut-off Λ imposed on the momentum transfers from individual collisions. $Q_{\perp 0} \sim (\hat{q}E)^{1/4}$ is any rough estimate of the total transverse momentum transfer during one bremsstrahlung formation time,[1] and ambiguities in that guess of $O(1)$ factors only affect the answer beyond leading-log order. C_s is the quadratic Casimir associated with the color representation of particle s. For QCD,

$$C_q = C_F = \tfrac{4}{3}, \qquad C_g = C_A = 3. \quad (3.19)$$

The $P_{a\to b}$ are the usual Dokshitzer-Gribov-Lipatov-Alterelli-Parisi (DGLAP) splitting functions

$$P_{q\to g}(x) = C_F \frac{[1+(1-x)^2]}{x}, \quad (3.20)$$

$$P_{g\to g}(x) = C_A \frac{[1+x^4+(1-x)^4]}{x(1-x)}, \quad (3.21)$$

$$P_{g\to q}(x) = N_f t_F [x^2 + (1-x)^2], \quad (3.22)$$

where N_f is the number of quark flavors and t_s is the trace normalization of color generators T^a defined by $\mathrm{tr}(T_s^a T_s^b) = t_s\,\delta^{ab}$, with

$$t_F = \tfrac{1}{2}, \qquad t_A = C_A, \quad (3.23)$$

The UV-regulated $\hat{\bar{q}}(\Lambda)$ discussed above is given in weak coupling by

$$\hat{q}_s(\Lambda) \equiv C_s \hat{\bar{q}}(\Lambda) = \int_{q_\perp < \Lambda} d^2 q_\perp \frac{d\Gamma_{el,s}}{d^2 q_\perp} q_\perp^2, \quad (3.24)$$

where Γ_{el} is the rate of elastic 2→2 scattering of a high-energy particle of type s off of the plasma, and $\boldsymbol{q}_\perp$ is the transverse momentum transfer in an individual such collision. The only dependence on the type of high-energy particle is a factor of C_s, which we have factored out of the definition of $\hat{\bar{q}}$.

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[1] Here is a lightning review. By the uncertainty principle, the formation time t_f is of order $1/\Delta E$, where ΔE is the amount by which energy would be violated if an on-shell high-energy particle split by bremsstrahlung (or pair production) in isolation. For hard bremsstrahlung, $\Delta E \sim Q_\perp^2/E$ in the high-energy limit, where $Q_\perp$ is the transverse momentum of the final particles, and so $t_f \sim E/Q_\perp^2$. The $Q_\perp$ must be supplied by collisions with the plasma. By definition of $\hat{q}$, the amount of $Q_\perp$ picked up in one formation time is given by $Q_\perp^2 \sim \hat{q}t_f$. Putting the last two equations together gives $t_f \sim (E/\hat{q})^{1/2}$ and $Q_\perp \sim (\hat{q}E)^{1/4}$. (For a review, see, for example, Sec. 3.2 of Ref. [13] together with Ref. [14].)

The weak-coupling result for $\hat{\bar{q}}(\Lambda)$ is slightly different depending on the precise range of Λ. At leading-log order in coupling,[2]

$$\hat{\bar{q}}(\Lambda) \simeq \alpha T m_D^2 \ln\left(\frac{T^2}{m_D^2}\right) + 4\pi\alpha^2 \mathcal{N} \ln\left(\frac{\Lambda^2}{T^2}\right) \tag{3.25}$$

if $\Lambda \gtrsim T$, where the second term dominates in the limit of large Λ; and

$$\hat{\bar{q}}(\Lambda) \simeq \alpha T m_D^2 \ln\left(\frac{\Lambda^2}{m_D^2}\right) \tag{3.26}$$

if $m_D \ll \Lambda \lesssim T$. Here m_D is the Debye mass, given by

$$m_D^2 = \tfrac{1}{3}(t_A + N_f t_F)g^2 T^2 = (1 + \tfrac{1}{6} N_f)g^2 T^2, \tag{3.27}$$

and $\mathcal{N}$ is the plasma particle density weighted by group factors, given by

$$\mathcal{N} = \tfrac{1}{3}(t_A + \tfrac{3}{2} N_f t_F)\frac{\zeta(3)}{\zeta(2)} T^3 = (1 + \tfrac{1}{4} N_f)\frac{\zeta(3)}{\zeta(2)} T^3, \tag{3.28}$$

where $\zeta(s)$ is the Riemann zeta function.

2. Leading log stopping distances

Substituting the leading-log bremsstrahlung and pair production rates into the general leading-log formulas (3.8) and (3.16) gives us leading-log stopping distances. For the quark number stopping distance (3.8),

$$A_q = \frac{\alpha}{2\pi\sqrt{2}}[\hat{\bar{q}}(Q_{\perp 0})]^{1/2} \int_0^1 dx \, P_{q\to g}(x) \left[\frac{C_A + (2C_F - C_A)x^2 + C_A(1-x)^2}{x(1-x)}\right]^{1/2}$$
$$\times \left[1 - (1-x)^{1/2}\right], \quad (3.29)$$

Now take $Q_{\perp 0} \sim (\hat{q}E)^{1/4} \sim (\alpha^2 T^3 E)^{1/4}$. If $Q_{\perp 0} \ll T$, then (3.26) gives

$$\hat{\bar{q}}(Q_{\perp 0}) \simeq \alpha T m_D^2 \ln\left(\frac{Q_{\perp 0}^2}{m_D^2}\right) \simeq \tfrac{1}{2}\alpha T m_D^2 \ln\left(\frac{\alpha^2 T^3 E}{m_D^4}\right) \simeq \tfrac{1}{2}\alpha T m_D^2 \ln\left(\frac{E}{T}\right) \tag{3.30}$$

at leading-log order. Note that the factors of coupling cancel out in the logarithm. If $Q_{\perp 0} \gg T$, and if we keep track of logarithms of coupling as well as logarithms of energy, then

$$\hat{\bar{q}}(Q_{\perp 0}) \simeq \alpha T m_D^2 \ln\left(\frac{T^2}{m_D^2}\right) + 4\pi\alpha^2 \mathcal{N} \ln\left(\frac{Q_{\perp 0}^2}{T^2}\right)$$
$$\simeq 4\pi\alpha^2 \mathcal{N} \left[\ln\left(\frac{Q_{\perp 0}^2}{m_D^2}\right) + \frac{c}{2}\ln\left(\frac{T^2}{m_D^2}\right)\right]$$
$$\simeq 2\pi\alpha^2 \mathcal{N} \left[\ln\left(\frac{E}{T}\right) + c\ln\left(\frac{1}{\alpha}\right)\right]$$
$$\simeq 2\pi\alpha^2 \mathcal{N} \ln\left(\frac{E}{\alpha^c T}\right) \tag{3.31}$$

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[2] For a discussion of the perturbative result for $\hat{q}(\Lambda)$ using this notation, see Ref. [12].

with

$$c = 2 \left[\frac{T m_{\mathrm{D}}^2}{4\pi \alpha \mathcal{N}} - 1 \right]. \tag{3.32}$$

In what follows, it will be convenient to define the dimensionless numbers

$$\hat{m}_{\mathrm{D}}^2 \equiv \frac{m_{\mathrm{D}}^2}{\alpha T^2} = \frac{4\pi}{3}(t_{\mathrm{A}} + N_{\mathrm{f}} t_{\mathrm{F}}), \tag{3.33}$$

$$\hat{\mathcal{N}} \equiv \frac{4\pi \mathcal{N}}{T^3} = \frac{4\pi}{3}(t_{\mathrm{A}} + \tfrac{3}{2} N_{\mathrm{f}} t_{\mathrm{F}}) \frac{\zeta(3)}{\zeta(2)}. \tag{3.34}$$

We can now summarize the leading-log result as being given by (1.3) with

$$a = \frac{\hat{\mathcal{Q}}^{1/2}}{4\pi} \int_0^1 dx \, P_{\mathrm{q} \to \mathrm{g}}(x) \left[\frac{C_{\mathrm{A}} + (2C_{\mathrm{F}} - C_{\mathrm{A}})x^2 + C_{\mathrm{A}}(1-x)^2}{x(1-x)} \right]^{1/2} \left[1 - (1-x)^{1/2} \right], \quad (3.35)$$

$$\hat{\mathcal{Q}} \equiv \begin{cases} \hat{\mathcal{N}}, & Q_\perp \gg T; \\ \hat{m}_{\mathrm{D}}^2, & Q_\perp \ll T, \end{cases} \tag{3.36}$$

and

$$c = \begin{cases} 2\left(\frac{\hat{m}_{\mathrm{D}}^2}{\hat{\mathcal{N}}} - 1 \right), & Q_\perp \gg T; \\ 0, & Q_\perp \ll T. \end{cases} \tag{3.37}$$

Plugging in QCD group factors and doing the integral (3.35) numerically gives the results for a and c in Tables I and II with

$$a = 0.55634 \, \hat{\mathcal{Q}}^{1/2} \tag{3.38}$$

The result listed for one-flavor QED corresponds to setting $C_{\mathrm{A}} = t_{\mathrm{A}} = 0$ and $C_{\mathrm{F}} = t_{\mathrm{F}} = 1$.

A similar analysis of the energy stopping distances of (3.15) gives

$$\ell_s^{(\mathrm{e})} \simeq \frac{1}{a_s^{(\mathrm{e})} \alpha_{\mathrm{s}}^2 T} \sqrt{\frac{E}{TL}} \tag{3.39}$$

where, at leading-log order,

$$L \simeq \ln \left(\frac{E}{\alpha_{\mathrm{s}}^c T} \right) \tag{3.40}$$

as before, with the same value (3.37) of c. The coefficients $a_s^{(\mathrm{e})}$, however, are given by

$$a_{\mathrm{g}}^{(\mathrm{e})} = \frac{\hat{\mathcal{Q}}^{1/2}}{4\pi} \frac{(m_{\mathrm{gg}} m_{\mathrm{qq}} - m_{\mathrm{gq}} m_{\mathrm{qg}})}{m_{\mathrm{qq}} - m_{\mathrm{gq}}}, \qquad a_{\mathrm{q}}^{(\mathrm{e})} = \frac{\hat{\mathcal{Q}}^{1/2}}{4\pi} \frac{(m_{\mathrm{gg}} m_{\mathrm{qq}} - m_{\mathrm{gq}} m_{\mathrm{qg}})}{m_{\mathrm{gg}} - m_{\mathrm{qg}}}, \tag{3.41}$$

where

$$m_{\mathrm{gg}} = \int_0^1 dx \left\{ \tfrac{1}{2} P_{\mathrm{g} \to \mathrm{g}}(x) \left[\frac{C_{\mathrm{A}} + C_{\mathrm{A}} x^2 + C_{\mathrm{A}} (1-x)^2}{x(1-x)} \right]^{1/2} \left[1 - x^{3/2} - (1-x)^{3/2} \right] \right.$$

$$\left. + P_{\mathrm{g} \to \mathrm{q}}(x) \left[\frac{(2C_{\mathrm{F}} - C_{\mathrm{A}}) + C_{\mathrm{A}} x^2 + C_{\mathrm{A}} (1-x)^2}{x(1-x)} \right]^{1/2} \right\}, \tag{3.42}$$

$$m_{\mathrm{gq}} = \int_0^1 dx\, P_{\mathrm{g} \to \mathrm{q}}(x) \left[\frac{(2C_{\mathrm{F}} - C_{\mathrm{A}}) + C_{\mathrm{A}} x^2 + C_{\mathrm{A}} (1-x)^2}{x(1-x)} \right]^{1/2} \left[-x^{3/2} - (1-x)^{3/2} \right], \tag{3.43}$$

$$m_{\mathrm{qg}} = \int_0^1 dx\, P_{\mathrm{q} \to \mathrm{g}}(x) \left[\frac{C_{\mathrm{A}} + (2C_{\mathrm{F}} - C_{\mathrm{A}}) x^2 + C_{\mathrm{A}} (1-x)^2}{x(1-x)} \right]^{1/2} \left[-x^{3/2} \right], \tag{3.44}$$

$$m_{\mathrm{qq}} = \int_0^1 dx\, P_{\mathrm{q} \to \mathrm{g}}(x) \left[\frac{C_{\mathrm{A}} + (2C_{\mathrm{F}} - C_{\mathrm{A}}) x^2 + C_{\mathrm{A}} (1-x)^2}{x(1-x)} \right]^{1/2} \left[1 - (1-x)^{3/2} \right]. \tag{3.45}$$

Results are displayed in Tables I and II.

D.   Quark mass thresholds

In this paper, N_{f} is the number of effectively massless quark species. Here we comment on what counts as a massless quark for the purposes of our analysis. Parametrically, the condition to ignore the mass of a high-energy quark in a splitting process $q \to qg$ or $g \to q\bar{q}$ is[3]

$$M \ll Q_\perp \sim (\hat{q} E)^{1/4}. \tag{3.46}$$

So we can use our formulas to describe the quark number stopping time of such a quark, or we can compute energy stopping times provided we take N_{f} in the DGLAP splitting function (3.22) to be the number of quark species that satisfy (3.46). All other factors of N_{f} in this paper refer to the number of quark species in the plasma, which should be taken as the number satisfying $M \ll T$. The specific results in tables I and II are given for the case that these two N_{f} values are the same.

IV.   NLLO ANALYSIS OF QUARK NUMBER STOPPING

Now we extend the analysis of the previous section to next-to-leading logarithmic order for the case of the quark number stopping distance. Our goal is to determine the coefficient b in (1.3).

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[3] See footnote 1. For hard bremsstrahlung, the quark mass will give a contribution of order M^2/E to $\Delta E \sim Q_\perp^2/E$. This effect of the mass is negligible when $M \ll Q_\perp$. Plugging in the massless result $Q_\perp \sim (\hat{q} E)^{1/4}$ makes this condition (3.46).

A. NLLO splitting rates

The NLLO result for splitting rates was computed in Refs. [11, 12] and corresponds to the leading-log result (3.17),

$$\frac{d\Gamma_{a\to bc}}{dx} = \frac{\alpha\,\mu_{a\to bc}^2\,P_{a\to b}(x)}{4\pi\sqrt{2}\,x(1-x)E}\,, \tag{4.1a}$$

with the formula (3.18) for $\mu_{a\to bc}^2$ replaced by

$$\mu_{a\to bc}^2 \simeq \left\{ 4x(1-x)E\left[(-C_a + C_b + C_c)\,\hat{\hat{q}}\left(\xi^{1/2}\mu_{a\to bc}\right) + (C_a - C_b + C_c)x^2\,\hat{\hat{q}}\left(\frac{\xi^{1/2}\mu_{a\to bc}}{x}\right)\right.\right.$$

$$\left.\left. + (C_a + C_b - C_c)(1-x)^2\,\hat{\hat{q}}\left(\frac{\xi^{1/2}\mu_{a\to bc}}{1-x}\right)\right]\right\}^{1/2}, \tag{4.1b}$$

where

$$\xi \equiv \exp(2 - \gamma_{\rm E} + \tfrac{\pi}{4}). \tag{4.2}$$

Numerically, equation (4.1b) may be solved self-consistently for $\mu_{a\to bc}$. Alternatively, one may solve it iteratively starting from an initial guess $\mu_{a\to bc} = Q_{\perp 0} \sim (\hat{q}E)^{1/4}$ on the right-hand side, then using the result as a refined guess, and so forth. This generates an explicit expansion of the solution in inverse powers of logarithms. In either case, the final result for the rate $d\Gamma/dx$ is only correct to NLLO. For the quark-number stopping distance, we only care about the $q \to gq$ splitting rate.

The explicit expansion of the rate (4.1a) in powers of inverse logarithms has the form

$$\frac{d\Gamma}{dx} = \alpha^2 T\left(\frac{T}{E}\right)^{1/2}\mathcal{A}(x)\,\ln^{1/2}(\kappa E)\left[1 + \frac{\tfrac{1}{2}\ln[\ln(\kappa E)] + \mathcal{B}(x)}{\ln(\kappa E)} + \cdots\right], \tag{4.3}$$

where κ is an arbitrary constant with dimensions of inverse energy. $\mathcal{B}(x)$ and higher-order coefficients in the expansion implicitly depend on κ in such a way that the rate is formally independent of the choice of κ — that is, the dependence on κ at any fixed order in the expansion is always a higher-order effect. However, in practice κ should be chosen to be parametrically of order $1/\alpha^c T$. (We will make a much more specific choice of κ later on.)

$\mathcal{A}(x)$ simply gives the leading-log term, which we will find notationally convenient to write as (specializing to $q \to gq$)

$$\mathcal{A}(x) = \frac{[\lambda(x)]^{1/2}}{4\pi x(1-x)}\,P_{q\to g}(x) \tag{4.4}$$

with

$$\lambda(x) \equiv \hat{\mathcal{Q}}x(1-x)\left[C_{\rm A} + (2C_{\rm F} - C_{\rm A})x^2 + C_{\rm A}(1-x)^2\right]. \tag{4.5}$$

For NLLO, we need better than the leading-log result (3.25) for $\hat{\hat{q}}(\Lambda)$. Complete results to leading order in powers of coupling are presented in Refs. [12, 15].[4] For the case $\Lambda \gg T$,

$$\hat{\hat{q}}(\Lambda) \simeq \alpha^2 T^3\left[\hat{m}_{\rm D}^2\ln\frac{\xi' T^2}{m_{\rm D}^2} + \hat{\mathcal{N}}\ln\frac{\Lambda^2}{\xi' T^2} - \tfrac{16}{\pi}(t_{\rm A}\sigma_+ + 2N_{\rm f}t_{\rm F}\sigma_-)\right] \tag{4.6}$$

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[4] Ref. [15] also calculated the next-order correction in powers of the coupling g, which is quite significant for any value of $\alpha_{\rm s}$ of possible interest to experiment.

where

$$\xi' \equiv 4 \exp\left(1 - 2\gamma_{\mathrm{E}}\right),$$ (4.7)

and

$$\sigma_+ \equiv \sum_{k=1}^{\infty} \frac{1}{k^3} \ln[(k-1)!] \simeq 0.3860438,$$ (4.8)

$$\sigma_- \equiv \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k^3} \ln[(k-1)!] \simeq 0.0112168.$$ (4.9)

For the case $\Lambda \ll T$, the leading-order answer for $\hat{\bar{q}}$ is in fact the same as (3.26)

$$\hat{\bar{q}}(\Lambda) \simeq \alpha T m_{\mathrm{D}}^2 \ln\left(\frac{\Lambda^2}{m_{\mathrm{D}}^2}\right).$$ (4.10)

Combining these expressions for $\hat{\bar{q}}(\Lambda)$ with (4.1) and (4.3) gives

$$\mathcal{B}(x) = \tfrac{1}{2}\ln\left(\frac{1}{\alpha^c T \kappa}\right) + \mathcal{B}_1(x) + \mathcal{B}_2(x),$$ (4.11)

with

$$\mathcal{B}_1(x) = \tfrac{1}{2}\ln[2\,\lambda(x)] - 2\,\frac{[(2C_{\mathrm{F}} - C_{\mathrm{A}})x^2 \ln x + C_{\mathrm{A}}(1-x)^2 \ln(1-x)]}{[C_{\mathrm{A}} + (2C_{\mathrm{F}} - C_{\mathrm{A}})x^2 + C_{\mathrm{A}}(1-x)^2]},$$ (4.12a)

$$\mathcal{B}_2(x) = \begin{cases} \ln\frac{\xi}{\xi'} + \frac{1}{\mathcal{N}}\left[\hat{m}_{\mathrm{D}}^2 \ln\frac{\xi'}{\hat{m}_{\mathrm{D}}^2} - \frac{16}{\pi}(t_{\mathrm{A}}\sigma_+ + 2N_{\mathrm{f}}t_{\mathrm{F}}\sigma_-)\right], & Q_\perp \gg T; \\ \ln\frac{\xi}{\hat{m}_{\mathrm{D}}^2}, & Q_\perp \ll T. \end{cases}$$ (4.12b)

B.   NLLO stopping distance

Corresponding to the expansion (4.3) of the rate, one may now look for a solution to the quark number stopping distance equation (3.5) in the form

$$\ell_{\mathrm{q}}(E) = \frac{E^{1/2}}{A \ln^{1/2}(\kappa E)}\left[1 - \frac{\tfrac{1}{2}\ln[\ln(\kappa E)] + B}{\ln(\kappa E)} + \cdots\right].$$ (4.13)

The internal minus sign in this formula compared to (4.3) arises because the length is related to the inverse of the rate. Plugging the ansatz (4.13) and the NLLO rate (4.3) into (3.5) determines the same $A = a\alpha^2 T^{3/2}$ as in the leading-log calculation, with

$$a = \int_0^1 dx\, \mathcal{A}(x)\left[1 - (1-x)^{1/2}\right].$$ (4.14)

It also determines

$$\begin{aligned} B &= \frac{1}{a}\int_0^1 dx\, \mathcal{A}(x)\left\{\mathcal{B}(x)\left[1 - (1-x)^{1/2}\right] + \tfrac{1}{2}(1-x)^{1/2}\ln(1-x)\right\} \\ &= -\tfrac{1}{2}\ln(\alpha^c T \kappa) + \tfrac{1}{2}\beta \end{aligned}$$ (4.15)

with

$$\beta \equiv \frac{1}{a} \int_0^1 dx\, \mathcal{A}(x) \left\{ 2[\mathcal{B}_1(x) + \mathcal{B}_2(x)] \left[1 - (1-x)^{1/2}\right] + (1-x)^{1/2} \ln(1-x) \right\}. \qquad (4.16)$$

As mentioned earlier, the precise choice of κ is formally arbitrary, and the effect of changing κ by a multiplicative factor of order one does not affect the NLLO result except by corrections that are yet higher order in inverse logarithms. However, as a practical matter, it is useful to have some definite prescription for choosing a sensible value for κ. We will use the fastest apparent convergence (FAC) prescription, which in this context is to choose κ such that the NLLO correction in (4.13) vanishes:

$$\tfrac{1}{2}\ln[\ln(\kappa E)] - \tfrac{1}{2}\ln(\alpha^c T \kappa) + \tfrac{1}{2}\beta = 0, \qquad (4.17)$$

and so

$$\ln(\kappa E) = \ln\left(\frac{e^\beta E \ln(\kappa E)}{\alpha^c T}\right). \qquad (4.18)$$

This corresponds to (1.3b) with the identification $L = \ln(\kappa E)$ and

$$b = e^\beta. \qquad (4.19)$$

The combination of equations (4.12), (4.16), and (4.19) is our final result for the NLLO coefficient b in (1.3). Evaluating the integrals numerically gives the tabulated results for b in Tables I and II.

V. A NUMERICAL TEST OF THE RESULT

In this paper, we have derived a simple NLLO formula (1.3) for the quark number stopping distance of a high-energy particle. This result relies on an expansion in powers of inverse logarithms, which parametrically requires $\ln(E/T) \gg 1$, and one may wonder just how large E has to be before this assumption becomes reasonable. As an alternative, one could imagine dropping the assumption that logarithms are large and instead computing the stopping distance using the full weak-coupling result for the bremsstrahlung rate, at leading order in α_s [16, 17, 18, 19]. The disadvantage is that one must then do (somewhat complicated) numerics—we know of no simple formula for the stopping distance unless one assumes $\ln(E/T) \gg 1$. In this section, we perform such a numerical analysis simply as a check that our result (1.3) is indeed correct at sufficiently large E.

The numerics consist of Monte Carlo evolution of a large sample of quarks with energy E, using the full weak-coupling result[5] for the bremsstrahlung rate $d\Gamma/dx$ to randomly determine whether each quark loses energy xE in each small time step Δt. We will focus on the case $E \gg T$ and so may (for simplicity) ignore other mechanisms of energy loss, such as collisional energy loss. For the same reason, we only consider energy loss and will

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[5] Specifically, we take $d\Gamma_{a\to bc}/dx = (2\pi)^3 \gamma_{a\to bc}/E\nu_a$ where the $\gamma_{a\to bc}$ are the splitting functions of Refs. [17, 18] and the number ν_a of spin+color degrees of freedom is 6 for a quark or anti-quark and 16 for a gluon. We do not include any final-state Bose enhancement or Fermi blocking factors for the final-state particles b and c: In the limit that their energies are large compared to T, no such factors are necessary.

ignore processes which can increase the quark energy (such as inverse bremsstrahlung). Also for the sake of simplicity, we will formally focus on the $Q_\perp \ll T$ case (parametrically $E \ll T/\alpha_s^2 \ln(\alpha_s^{-1})$), which is the approximation implicitly used in previous numerical calculations of the bremsstrahlung rate for weak coupling.[6] In the numerics, we will call a quark "stopped" when its energy drops below T, since our approximations no longer make sense then and since a real quark would equilibrate once its energy dropped to be of order T. Whether one defines "stopped" as $E < T$ or $E < T/2$ or $E < 2T$ will only affect the stopping distance by a relative correction that is parametrically of order $\sqrt{T/E}$ (up to logarithms) and so is unimportant when $E \gg T$.

In each time step, we let each quark bremsstrahlung with probability $\Gamma \Delta t$. When bremsstrahlung occurs, we randomly choose the momentum fraction of the bremsstrahlung gluon according to the probability distribution $\Gamma^{-1} d\Gamma/dx$. This is not completely straightforward because the total bremsstrahlung rate Γ has a small x divergence. Radiation of small x gluons is inefficient for energy loss, and so the final result for the stopping distance should not be sensitive to this divergence. In our numerics, we simply sidestep such divergences by restricting the range of x values to $\epsilon_1 \le x \le 1-\epsilon_2$, where ϵ_1 and ϵ_2 are small, and then we check that our numerical results converge as we make ϵ_1 and ϵ_2 (and Δt) smaller and smaller.

A comparison of the full numeric result with the NLLO approximation (1.3) is shown in Fig. 2 as a function of E/T for 3-flavor QCD. The plot shows the size of the relative error $\epsilon \equiv 1 - t_{\mathrm{NLLO}}/t_{\mathrm{numeric}}$ that the NLLO approximation makes compared to the numeric simulation. The plot extends to absurdly large values of E/T because the purpose of this plot is to check our NLLO analysis. We can see that the NLLO result indeed approaches the full weak-coupling result at very large E. Moreover, it does not do too badly even at smaller values of E/T that are only moderately large: The difference is $\lesssim 30\%$ for $E \gtrsim 6T$. One should not take this too seriously, however, since at $E \sim 6T$ the various approximations we made earlier in this section are in doubt. We do not show any comparison for even smaller E/T simply because the NLLO result (1.3), which implements the FAC prescription, ceases to exist: There is no solution to (1.3b) for L when $E/T \le \alpha_s^c e/b$, which in the present case is $E/T \le 5.74$. At exactly $E/T = \alpha_s^c e/b$, the logarithm L becomes 1, and so the idea of a large-logarithm approximation has certainly broken down then.

As a further check of our NLLO calculation, we plot $\epsilon \ln^2(E/T)$ versus $y \equiv 1/\ln(E/T)$ in Fig. 3. If the NNLO contribution was incorrect, the result would diverge as $y \to 0$.

Finally, just to give some idea of the distribution of stopping lengths in the high-energy limit, we plot the distribution from our highest-energy numerical simulation in Fig. 4. For some quantities of interest in the subject of energy loss (such as the medium effect on energy loss from passage through a thin medium), there are slowly falling tails to distributions which cause average values of those quantities to be unrepresentative of typical events. That is not the case here for the stopping length: the average coincides with the location of the peak of the distribution.

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[6] Specifically, numerical calculations such as Refs. [18, 19] have used the formula $\mathcal{A}(q_\perp) = m_D^2 T/q_\perp^2 (q_\perp^2 + m_D^2)$ in the notation of Ref. [18]. ($\mathcal{A}(q_\perp)$ corresponds to $(2\pi/g)^2 d\Gamma_{\mathrm{el}}/d^2q_\perp$ in this paper's notation and $T\,C(q_\perp)$ in the notation of Ref. [19].) This formula is correct only when $Q_\perp \ll T$. As noted in Ref. [11], for 3-flavor QCD, use of this formula in the $Q_\perp \gg T$ case should produce an error of at most 15%.
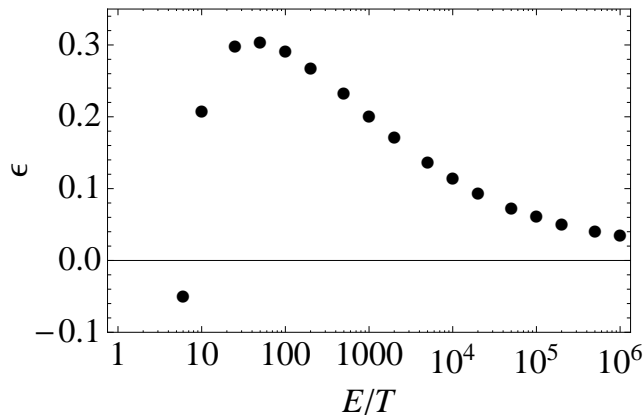
FIG. 2: The relative difference $\epsilon \equiv 1 - t_{\mathrm{NLLO}}/t_{\mathrm{numeric}}$ between (i) the NLLO quark number stopping distance (1.3) and (ii) the quark number stopping distance t_{numeric} computed through simulation using the full weak-coupling result for the bremsstrahlung rate. The slight scatter of points is due to the statistics of simulating only a finite number of particles (in this case, 10^5 particles).
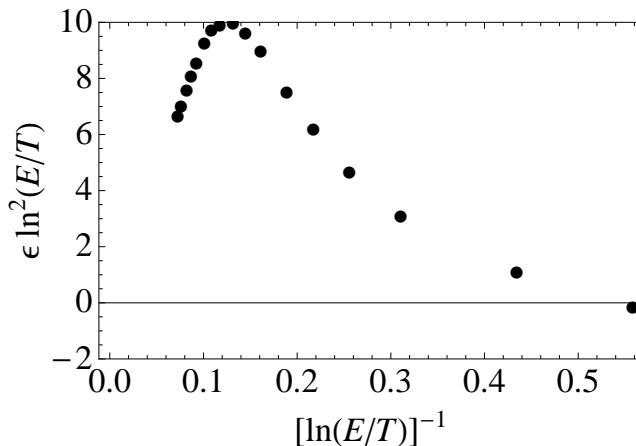

FIG. 3: The results of Fig. 2 plotted with axes convenient for checking the correctness of the NLLO result at very large E.

VI. RUNNING COUPLING

Our result (1.3) formally assumes that $\alpha \ln(E/T) \ll 1$ and so ignores, for instance, running of the coupling α. One can plausibly accommodate running in the bremsstrahlung rate, and so the stopping distance, by the prescriptions reviewed in Ref. [12].[7] The α^2 in the denominator of (1.3) comes from (i) the overall factor of α in (3.17) associated with the emission vertex for the bremsstrahlung gluon, and (ii) the factor μ^2 in (3.17), which is proportional to $\sqrt{\hat{q}} \propto \sqrt{\alpha^2}$. As reviewed in Ref. [12], the emission vertex factor should

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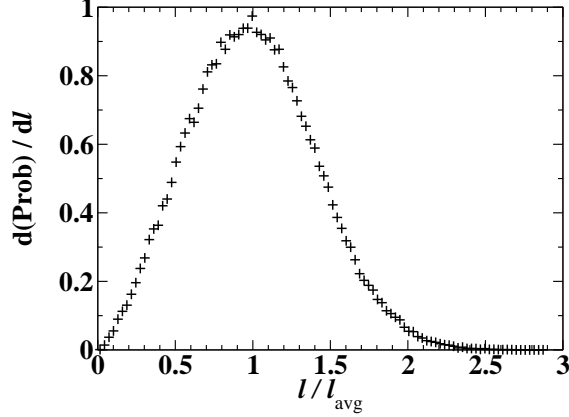[7] See also the earlier discussion in Refs. [3, 11, 20, 21].

FIG. 4: Simulation results for the probability distribution for quarks to lose their energy through bremsstrahlung in a distance ℓ, as function of ℓ/ℓ_{avg}, where ℓ_{avg} is the average. The particular results shown are for $E/T = 10^6$, for which ℓ_{avg} agrees with the NLLO stopping length (1.3) to within 4%.

plausibly be evaluated at the scale $Q_\perp \sim \mu \sim (\hat{q}E)^{1/4}$, and the running of α in $\hat{q}$ replaces[8]

$$\hat{q}(\Lambda) \propto \alpha^2 \quad \longrightarrow \quad \hat{q}(\Lambda) \propto \alpha(\Lambda)\,\alpha(m_{\mathrm{D}}). \tag{6.1}$$

Since we needed $\hat{q}$ at $\Lambda \sim Q_\perp$, the effect of both these replacements is to replace (1.3) by

$$\ell_{\mathrm{stop,q}} \longrightarrow \frac{1}{a\,\alpha^{3/2}(Q_\perp)\,\alpha^{1/2}(m_{\mathrm{D}})\,T}\sqrt{\frac{E}{TL}} \tag{6.2}$$

with $Q_\perp \sim (\hat{q}E)^{1/4}$. The origin of the α^c in (1.3b) is the difference between the scales m_{D} and T in the arguments of the logarithms of (4.6), and so that α^c should simply be $[\alpha(m_{\mathrm{D}})]^c$. Throughout this discussion, we do not distinguish between $\alpha(T)$ and $\alpha(m_{\mathrm{D}})$ since their difference is small when α is small.

The result (6.1) only accounts for $\alpha \ln(E/T)$ corrections associated with running of the coupling. We are not sure whether there might be other corrections of comparable size, which is why our introduction is only so bold as to claim a definite result in the formal limit $\alpha \ln(E/T) \ll 1$.

Acknowledgments

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[8] This is a slight simplification. This nice compact formulation of the prescription only works if there are no quarks masses between m_{D} and Λ.

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